Learning Together Through Collaborative Research: The Case of Proof in Secondary Mathematics

Michelle Cirillo
University of Delaware
Department of Mathematical Sciences

Jennifer Reed
Odyssey Charter School
Wilmington, DE

MSU Mathematics Education Colloquium
November 20, 2019
Research on Proof in School Mathematics

• Proof is important – the “guts of mathematics” (Wu, 1996).

BUT

• Proof is challenging for teachers to teach (e.g., Knuth, 2002, Cirillo, 2009; 2014).

• Proof is difficult for students to learn (Senk, 1985; McCrone & Martin, 2004).
How well do students write geometry proofs?

Sharon Senk (1985)
Senk’s Recommendations

We must immediately look for more effective ways to teach proof in geometry. We should:

– Pay special attention to teaching students to start a chain of reasoning;
– Place greater emphasis on the meaning of proof than we do currently; and
– Teach students how, why, and when they can transform a diagram in a proof.
Three Major Difficulties in the Learning of Demonstrative Geometry

Rolland R. Smith (1940)
“Three Serious Learning Difficulties”

• Lack of familiarity with geometric figures

• Not sensing the meaning of the if-then relationship

• Inadequate understanding of the meaning of proof
Students’ Difficulties with Proof in Geometry

• “In summary, we have seen that students are extremely unsuccessful with formal proof in geometry.”
  
  (Clements & Battista, 1992)

• “The teaching of mathematical proof appears to be a failure in almost all countries.”

  (Hershkowitz et al., 2002, p. 675)
Calls for Additional Research...

• “The mandate to involve students in proving is likely to be met with the development of tools and norms that teachers can use to enable students to prove and to demonstrate that they are indeed proving.”
  (Herbst, 2002, p. 200)

• “…research is needed to understand the conditions in which teachers work and how those conditions impact the mathematical work that teachers can sustain”
  (Herbst, 2006, p. 314)
Timeline of Progress

Smith (1940)

Senk (1985)

Cirillo (2020)
Three Studies

• 2005-2008: Longitudinal Dissertation Study

• 2010-2013: The Geometry Proof Project

• 2015-2020: Proof in Secondary Classrooms: Decomposing a Central Mathematical Practice (i.e., The PISC Project)
STUDY 1: THE CASE OF MATT
Matt

You can't teach somebody how to do a proof....I mean if a student's really gonna do a mathematical proof, you look at the problem and you either see how you do it or you don't.
Textbook Examples

• Reasoning with Properties from Algebra

**EXAMPLE 2  Writing Reasons**

Solve $55z - 3(9z + 12) = -64$ and write a reason for each step.

**SOLUTION**

\[
\begin{align*}
55z - 3(9z + 12) &= -64 & \text{Given} \\
55z - 27z - 36 &= -64 & \text{Distributive property} \\
28z - 36 &= -64 & \text{Simplify.} \\
28z &= -28 & \text{Addition property of equality} \\
z &= -1 & \text{Division property of equality}
\end{align*}
\]
Textbook Examples

- Proving Statements about Segments

**Example 1**  
Symmetric Property of Segment Congruence

You can prove the Symmetric Property of Segment Congruence as follows.

**GIVEN** \( \overline{PQ} \cong \overline{XY} \)

**PROVE** \( \overline{XY} \cong \overline{PQ} \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \overline{PQ} \cong \overline{XY} )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \overline{PQ} = \overline{XY} )</td>
<td>2. Definition of congruent segments</td>
</tr>
<tr>
<td>3. ( \overline{XY} = \overline{PQ} )</td>
<td>3. Symmetric property of equality</td>
</tr>
<tr>
<td>4. ( \overline{XY} \cong \overline{PQ} )</td>
<td>4. Definition of congruent segments</td>
</tr>
</tbody>
</table>
The Case of Matt: Overall Findings

• Despite strong content knowledge and a good teacher prep program, Matt was at a loss for teaching proof beyond show-and-tell.
• Matt wanted to teach “real math,” not just show students completed Theorems in the boxes in his textbook.
• Matt’s focus shifted from getting through the required theorems to attempting to teach students to prove.
STUDY 2: THE CASE OF MIKE
Mike, High School Geometry Teacher

- 8 years of experience at start of project
- Mathematics and Science background
- Conventional Prentice Hall *Geometry* textbook
- Private boys’ school
- Described students as motivated, curious, confident, intelligent, and affluent
Mike Began Proof with Triangle Congruence

1. **GIVEN:** $\angle A \cong \angle D$, $\angle B \cong \angle E$
   
   $\overline{BC} \cong \overline{EC}$

   **PROVE:** $\triangle ABC \cong \triangle DEC$
VIDEO REMOVED DUE TO HUMAN SUBJECTS’ PERMISSIONS
BACK TO MATT FOR A BRIEF MOMENT...
“On Friday the students will begin constructing their own deductive proofs. Unfortunately, there is no good way, in my opinion, to ‘teach’ proofs. Students simply have to do them – like learning to swim by drowning.”

“Ok, there's only so many of these that I can do with us together. I just kind of, got to keep throwing you in the deep end. Letting you thrash around for awhile. And then throw you a floaty. Haul you back out and then throw you back in. Alright?”

(Cirillo, 2008)
BACK TO MIKE...
Things I need to know:

- How do I know what steps to write?
- How do I know what order the steps are in?
- Argh! I don’t even know where to start!!!
- How big should I make the T?
- What reasons am I allowed to use?
- How many steps do I need to write?
What makes teaching proof in geometry so tough?

- Curriculum
- Student Readiness
- Lack of recommendations for scaffolding the introduction to proof (i.e., understanding of the “shallow end” of the proof pool)
• What is going on for *students* when we introduce proof?
Perpendicular lines intersect to form right angles.

No Given? What can we assume from a diagram?

Given: \( \overline{JN} \) bisects \( \overline{KM} \)

Prove: \( \angle KJL \cong \angle MNL \)

\( \overline{JK} \perp \overline{KM} \) and \( \overline{MN} \perp \overline{KM} \) (Given)

\( \angle K \) and \( \angle M \) are right angles (Definition of Perpendicular Lines)

\( \angle K \cong \angle M \) (Theorem: If two angles are right angles, then they are congruent.)

\( \overline{KL} \cong \overline{LM} \) (Definition of Midpoint)

\( \angle JLK \) and \( \angle NLM \) are vertical angles (Definition of Vertical Angles)

\( \angle JLK \cong \angle NLM \) (Theorem: If two angles are vertical angles, then they are congruent.)

\( \triangle KJL \cong \triangle MNL \) (ASA \( \cong \) ASA)

\( \angle KJL \cong \angle MNL \) (CPCTC)
A Proof

Given: \( \overline{JN} \) bisects \( \overline{KM} \)
\( \overline{JK} \perp \overline{KM} \)
\( \overline{MN} \perp \overline{KM} \)

Prove: \( \angle KJL \cong \angle MNL \)

1. \( \angle K \) and \( \angle M \) are right angles
   (Definition of Perpendicular Lines)

2. \( \angle K \cong \angle M \)
   (Theorem: If two angles are right angles, then they are congruent.)

3. \( \overline{KL} \cong \overline{LM} \)
   (Definition of Midpoint)

4. \( \angle JLK \) and \( \angle NLM \) are vertical angles
   (Definition of Vertical Angles)

5. \( \angle JLK \cong \angle NLM \)
   (Theorem: If two angles are vertical angles, then they are congruent.)

6. \( \triangle KJL \cong \triangle MNL \)
   (ASA \cong ASA)

7. \( \angle KJL \cong \angle MNL \)
   (CPCTC)
If there was a shallow end to teaching proof, what would it look like?
STUDY 3: THE PISC PROJECT
PISC Project Timeline

Planning Year: 2015-2016
Baseline Data Collection & Lesson Piloting: 2016-2017
Professional Development & Summer Lesson Study: Spring & Summer, 2017
Pilot Lessons (Core Teachers): 2017-2018
Pilot Lessons (Again) (Core Teachers): 2018-2019
Publication & Dissemination: 2019-2020

Used Senk & Usiskin’s assessments with Control and Experimental Groups
SOME CLASSROOM VIDEOS

Y1 → Y3 (ACTUALLY Y2, Y4 OF PISC)
Cut-and-Paste Proofs

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>1.</td>
</tr>
<tr>
<td>2.</td>
<td>2.</td>
</tr>
<tr>
<td>3.</td>
<td>3.</td>
</tr>
<tr>
<td>4.</td>
<td>4.</td>
</tr>
<tr>
<td>5.</td>
<td>5.</td>
</tr>
</tbody>
</table>
Year 1: First Day of Triangle Congruence Proof

What do you notice and wonder?
VIDEO REMOVED DUE TO HUMAN SUBJECTS’ PERMISSIONS
Year 1: First Day of Triangle Congruence Proof

What do you notice and wonder?
“I noticed a lot of really great things you guys were doing. **You remembered to put your Given information first and to put what you’re trying to prove last** and for the most part it looked like we had a lot of things in the correct order, **but some of you, I feel like just put them there because you knew they had to be there, but you didn’t really go through the steps in the correct kind of order.** So that’s what we’re going to work on today.”
Year 1: First Day of Triangle Congruence Proof

• No logic
• 52 minutes
• Unsure
  • “They said...”
Year 3: First Day of Triangle Congruence Proof

**Given:** $BD$ is the $\perp$ bisector of $AC$

**Prove:** $\triangle ABD \cong \triangle CBD$
Year 3: First Day of Triangle Congruence Proof

- Presentation of student work
- Modeling how to discuss and critique the reasoning of others
VIDEO REMOVED DUE TO HUMAN SUBJECTS’ PERMISSIONS
Year 3: First Day of Triangle Congruence Proof

What do you notice and wonder?
Year 3: First Day of Triangle Congruence Proof

- Confidence in content
- Student input on making the proof better
- Using true logic
Given: $FI \parallel HI$
- $FI$ bisects $JIH$ at $G$

Prove: $\triangle JFG \cong \triangle HIG$

---

Given: $FI \parallel HI$
- $FI$ bisects $JIH$ at $G$

Prove: $\triangle JFG \cong \triangle HIG$

\[ \triangle JFG \cong \triangle HIG \]

ASA \cong ASA

\[ \triangle JFG \cong \triangle HIG \]

AAS \cong AAS
Year 3: First Day of Triangle Congruence Proof

- Student presentation
- Various outcomes
- Creating opportunities for students to engage with another’s reasoning
VIDEO REMOVED DUE TO HUMAN SUBJECTS’ PERMISSIONS
Year 3: First Day of Triangle Congruence Proof

What do you notice and wonder?
Year 3: First Day of Triangle Congruence Proof

- Student-focused
- Various methods to solve
- Teacher discourse moves
PISC Project Timeline

WHAT HAPPENED IN HERE?

Video 1

Planning Year
Baseline Data Collection & Lesson Piloting
Professional Development & Summer Lesson Study
Pilot Lessons (Core Teachers)
Pilot Lessons (Again) (Core Teachers)
Publication & Dissemination

Phase I
2015-2016
Phase II
2016-2017
Phase III
Spring & Summer, 2017
Phase IV
2017-2018
Phase V
2018-2019
Phase VI
2019-2020
WHAT HAPPENED?
Between Year 1 and 3: What happened?

• Professional Development
  • Student Thinking
  • Summer Camp
    • Lesson Study
    • Debriefing
• Lessons and readings on Teacher Discourse Moves
• Teaching the Lessons
  • Video Recorded
  • Feedback
  • Daily Reflections
• Continuous PD
  • Met as a Group – Improved Lessons
  • Control Group
WHAT HAPPENED?
Welcome!

The Proof in Secondary Classrooms (PISC) project is a five-year CAREER grant funded by the National Science Foundation [PI: Michelle Cirillo]. PISC will develop an intervention to support the teaching and learning of proof in the context of geometry. This study takes as its premise that if we introduce proof, by first teaching students particular sub-goals of proof, such as how to draw a conclusion from a given statement and a definition, then students will be more successful with constructing proofs on their own.

PISC will draw on findings and artifacts from a previous 3-year study, funded by the Knowles Science Teaching Foundation, which considered the challenges of teaching proof in geometry. In this earlier study, classroom and interview data
## Geometry Proof Scaffold: A Pedagogical Framework for Teaching Proof

<table>
<thead>
<tr>
<th>Sub-Goals</th>
<th>Descriptions</th>
<th>Competencies</th>
</tr>
</thead>
</table>
| **Understanding Geometric Concepts** | This sub-goal highlights the importance of understanding the building blocks of geometry. | 1) Having an accurate “mental picture” of geometric concepts (i.e., having a concept image)  
2) Being able to verbally describe geometric concepts, ideally being fluent with one or more definitions of the concept (i.e., having or developing a concept definition)  
3) Determining examples and non-examples  
4) Understanding connections between classes of geometric objects, where they overlap, and how they are contained within other classes (i.e., understanding mathematical hierarchy) |
| **Coordinating Geometric Modalities** | This sub-goal highlights the ways in which the mathematics register evolves over a range of modalities. | 1) Translating between diagrams and symbolic notation  
2) Translating between language and diagram  
3) Translating between languages and mathematical notation |
| **Defining** | This sub-goal highlights the nature of definitions, their logical structure, how they are written, and how they are used. | 1) Writing a “good” definition (includes necessary and sufficient properties)  
2) Knowing definitions are not unique (i.e., geometric objects can have different definitions)  
3) Understanding how to write and use definitions or biconditionals  
4) Understanding that empirical reasoning can be used to develop a conjecture but it is not sufficient proof of the conjecture  
5) Being able to turn a conjecture into a testable conditional statement  
6) Seeking out counterexamples to test conjectures and knowing that only one counterexample is needed to disprove a conjecture |
| **Conjecturing** | This sub-goal recognizes that conjecturing is an important part of mathematics and proving. | 1) Recognizing a sub-argument as a branch of proof and how it fits into the larger proof  
2) Understanding what valid conclusions can be drawn from a given statement and how these make a sub-argument (i.e., knowing some commonly occurring sub-arguments)  
3) Understanding how to write a sub-argument using notation and acceptable language (where “acceptable” is typically determined by the teacher) |
| **Drawing Conclusions** | This sub-goal presents the idea of an open-ended task that leads to conclusions that can be drawn from given statements and/or a diagram. | 1) Understanding what can and cannot be assumed from a diagram  
2) Knowing what information and/or “given” information can be used to draw a conclusion from a statement about a mathematical object  
3) Using postulates, definitions, and theorems (or combinations thereof) to draw valid conclusions from given information |
| **Understanding Common Sub-arguments** | This sub-goal recognizes that there are common short sequences of statements and reasons that are used frequently in proofs and that these may appear relatively unchanged from one proof to the next. | 1) Understanding that a theorem is not a theorem until it has been proven  
2) Understanding that one cannot use the conclusions of the theorem itself to prove the conclusions of another theorem (i.e., avoiding circular reasoning)  
3) Understanding that theorems are mathematical statements that are easy to prove and that their structure is logical and consistent  
4) Understanding the connection between logic and a theorem, for example, how to write the contrapositive of a conditional statement |
| **Understanding Theorems** | This sub-goal highlights the nature of theorems, their logical structure, how they are written, and how they are used. | 1) Understanding and explaining a theorem statement to determine the hypothesis and conclusion, and, if needed, providing appropriate diagrams  
2) If applicable, marking a diagram that satisfies the hypothesis of a proof  
3) Writing a theorem in words in symbols and vice versa  
4) Understanding that a theorem is not a theorem until it has been proven  
5) Understanding that one cannot use the conclusions of the theorem itself to prove the conclusions of another theorem (i.e., avoiding circular reasoning)  
6) Understanding that theorems are mathematical statements that are easy to prove and that their structure is logical and consistent  
7) Understanding the connection between logic and a theorem, for example, how to write the contrapositive of a conditional statement |
| **Understanding the Nature of Proof** | This sub-goal highlights the nature of proof, proof structure, and how the laws of logic are applied. | 1) Understanding that the only way to sanction the truth of a conjecture is through deductive proof (rather than empirical reasoning)  
2) Exploring a pathway for constructing a proof (i.e., the problem-solving aspect of proving)  
3) Understanding that proofs are constructed using axioms, postulates, definitions, and theorems and that they follow the laws of logic |
| **Proofing by Calculation** | This sub-goal highlights the nature of proof, proof structure, and how the laws of logic are applied. | 1) Understanding that the only way to sanction the truth of a conjecture is through deductive proof (rather than empirical reasoning)  
2) Exploring a pathway for constructing a proof (i.e., the problem-solving aspect of proving)  
3) Understanding that proofs are constructed using axioms, postulates, definitions, and theorems and that they follow the laws of logic |

Decomposing Proof
Show-and-Tell vs. An On-Ramp

- Teacher Show-and-Tell
- On-Ramp
The Geometry Proof Scaffold

(i.e., the “GPS”)
The Geometry Proof Scaffold

- Diagrams
- Language
- Notation
The Geometry Proof Scaffold (i.e., the “GPS”)
Given: $\overline{BD}$ bisects $\triangle ABC$.
2. Which of the following statements could you conclude from the Given information and the figure?

Given: \( \overline{BD} \) bisects \( \angle ABC \)

A. \( \overline{BD} \perp \overline{AC} \)
B. \( \overline{AD} \cong \overline{CD} \)
C. \( \angle ABD \cong \angle CBD \)
D. \( \triangle ABC \) is Isosceles
E. All of the above
The Geometry Proof Scaffold
(i.e., the “GPS”)
PISC Research Questions

• How do teachers *introduce* proof in geometry?
• When engaging in lesson study based on introducing proof by first teaching particular sub-goals of proof, how do teacher respond to and execute the lesson plans?
• How do students respond to these lessons?
• How do students in the control and experimental groups think about proof and perform on a set of proof tasks?
2016-2019
(Y2 - Y4)

Data Collected

Assessments
- 1,550 Pre-Tests Administered (EGT)
- 1,278 Post-Tests Administered (SGT)

Interviews
- 24 Teacher Interviews
- 31 Student Interviews

Classroom Observations
- 294 Classroom Observations
PISC Curriculum
DID THE TREATMENT WORK?
Core Teachers Item Averages

Teacher 1

Teacher 2

Teacher 3

Teacher 4
What is the estimated impact of the PISC curriculum on students’ SGT scores?

Regression Analysis of SGT Scores by Treatment/Control Condition
After Controlling for EGT Score
What is the estimated impact of the PISC curriculum on students’ SGT scores (Year 1 vs. Year 3 only)?

<table>
<thead>
<tr>
<th>HLM Model Parameters</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Effects</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>6.73</td>
<td>2.39</td>
<td>0.0125</td>
</tr>
<tr>
<td>EGT NCE Score</td>
<td>0.44</td>
<td>0.03</td>
<td>&lt;.0001</td>
</tr>
<tr>
<td>8th Grade Indicator</td>
<td>11.20</td>
<td>3.71</td>
<td>0.0026</td>
</tr>
<tr>
<td>CORE Treatment</td>
<td>6.61</td>
<td>1.75</td>
<td>0.0002</td>
</tr>
<tr>
<td>Random Effects (residuals)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teacher</td>
<td>45.22</td>
<td>18.0</td>
<td>0.0060</td>
</tr>
<tr>
<td>Student</td>
<td>122.34</td>
<td>6.47</td>
<td>&lt;.0001</td>
</tr>
</tbody>
</table>

After controlling for grade level and EGT scores and restricting analyses to Year 1 and Year 3 data only, students in CORE classes scored 6.61 NCE points higher (ES = +.31 standard deviations) on the SGT (p<.001).

Gains made by students were significantly larger in classrooms using the PISC curriculum.
Michelle’s Reflections

• “Collaboration between researchers and school personnel provides integrated perspectives for addressing critical issues in mathematics teaching and learning” (NCTM, 2012, p. 1).

• Impact of Attending to Student Thinking

• Cannot do this kind of work alone or on campus
Jen’s Reflections

- Well worth the time and effort
- Confidence in content
- Better teaching overall
This material is based upon work supported by the National Science Foundation (NSF; Award #1453493, PI: Michelle Cirillo). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

Gratitude to Jen Hummer, Amanda Seiwell, Kelly Curtis, many undergraduate students, and the project teachers.

Email mcirillo@udel.edu for questions about or visit www.pisc.udel.edu for updates on the project.